Temporal Diffusion: From Microscopic Dynamics to Generalised Fokker–Planck and Fractional Equations

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Abstract The temporal Fokker–Planck equation (Boon et al. in J Stat Phys 3/4: 527, 2003) or propagation–dispersion equation was derived to describe diffusive processes with temporal dispersion rather than spatial dispersion as in classical diffusion. We present two generalizations of the temporal Fokker–Planck equation for the first passage distribution function $f_j(r, t)$ of a particle moving on a substrate with time delays τ_j. Both generalizations follow from the first visit recurrence relation. In the first case, the time delays depend on the local concentration, that is the time delay probability P_j is a functional of the particle distribution function and we show that when the functional dependence is of the power law type, $P_j \propto f_j^{\nu-1}$, the generalized Fokker–Planck equation exhibits a structure similar to that of the nonlinear spatial diffusion equation where the roles of space and time are reversed. In the second case, we consider the situation where the time delays are distributed according to a power law, $P_j \propto \tau_j^{-1-\alpha}$ (with $0 < \alpha < 2$), in which case we obtain a fractional propagation-dispersion equation which is the temporal analog of the fractional spatial diffusion equation (with space and time interchanged). The analysis shows how certain microscopic mechanisms can lead to non-Gaussian distributions and non-classical scaling exponents.

Keywords Nonlinear transport · Fokker–Planck equation · Fractional kinetics

1 Introduction

Typical spatial diffusion processes are formulated in the continuum limit by the convection-diffusion equation whose solution is a Gaussian centered at the most-likely position of a particle (a walker) moving at a constant velocity. Reciprocally there are situations in which,

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instead of asking where the walker would be after a given time (long with respect to the duration of an elementary time step), one addresses the the question as to how long it takes to reach a given point, at some large distance from the starting position (large compared to the unit length covered during the elementary time step). For a stochastic process, one then asks what is the distribution of times taken to reach that point and the problem can be described by a propagation-dispersion equation giving a Gaussian time distribution centered at the most likely time of arrival at the target point [1,2]. This characterizes classical time dispersion when the distribution originates from an Einstein type recurrence relation [3] for the probability $f(r, t)$ of finding the particle at position r at time t as briefly described in Sect. 2. A practical example is given in [4] which describes an experiment where small beads are dropped into a container filled with larger beads. The small beads, driven by gravity, diffuse through the array of larger beads and their collisions with the larger ones induce time delays in the downward motion. Measurement of the arrival times of the small beads at the end point of the container gives a Gaussian distribution, (see Fig. 9 in [4]) i.e. the signature of a temporal-dispersion process.

In many problems in physics, chemistry and biology, the complexity of the media where particles move renders the classical description giving a Gaussian distribution incomplete and the subject has been intensely considered in the literature from the viewpoint of spatial diffusion (see e.g. the reviews in [5] and [6] with a well documented list of references). Concomitantly, in these media processes are time delayed or accelerated because they exhibit a functional dependence on the local concentration or on the time delays, in which cases one expects deviations from the classical Gaussian distribution of time delays. However, besides classical first passage phenomena [7], temporal diffusive processes have been rather overlooked in the literature.

Here we start with the generalised recurrence relation where the waiting time probability is a functional of the distribution function $f(r, t)$ and in Sect. 3 we derive a generalized Fokker–Planck equation (GFPE) for the first passage distribution function $f(r, t)$. Using a scaling argument (Sect. 4) we obtain its solution which is shown to exhibit a narrowing of the temporal distribution, i.e. temporal localization. Alternatively in Sect. 5 we introduce a power law *ansatz* for the time delay probability and we obtain a description of the evolution of the time distribution in the form of a fractional temporal Fokker–Planck equation (FFPE). So it follows that the macroscopic evolution of the system is given by two complementary descriptions, the nonlinear temporal Fokker–Planck equation or the fractional temporal Fokker–Planck equation, depending on the basic mechanisms of the time delay processes.

2 Generalized Recurrence Relation

Consider a walker moving on a one-dimensional lattice whose sites are labeled by integers $l = 0, 1, 2, \ldots, n$. The distance between neighboring sites is denoted by δr. The clock is set at $t_0 = 0$ when the particle is at site $l = 0$ and its trajectory will intercept successively sites $l = 1, 2, 3, \ldots$ for the first time at times $t_1, t_2, t_3 \ldots$. The t_j's are integer multiples of the time step δt. While sites $0, 1, 2, 3, \ldots$ are equally spaced, the time differences between first visits, $t_{j+1} - t_j$, are (in general) not equally distributed, that is when the particle is at position $r - \delta r$ it will take τ_j time steps to move to the next neighboring site r. So if j is the random variable which corresponds to the number of steps required for the particle to reach position r for the first time at time t, given it was at $r - \delta r$ at time $t - \tau_j$, the probability $f(r, t)$ of finding the particle at position r for the first time at time t is expressed by the finite recurrence equation
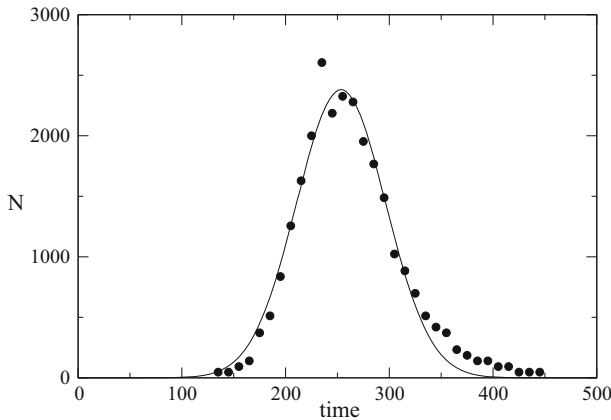
Fig. 1 Number of runners N versus finishing time (in min) for the New York marathon 1996 (data from *NY Marathon.nb* at http://www.tcsnycmarathon.org)

$$f(r, t) = \sum_{j=1}^{N} p_j \, f(r - \delta r, t - \tau_j), \tag{1}$$

where $\tau_j = j \, \delta t$. Here p_j is the time delay probability, i.e. the probability that it takes j time steps for the particle to move from site $r - \delta r$ to site r. In practice this probability can be obtained from a prescribed distribution or inferred from the physics of the specific problem considered or ideally extracted from possibly available data sets. Assuming the particles can reach any given position, the time-delay probabilities must be normalized as $\sum_{j=1}^{N} p_j = 1$ which is consistent with the normalization of the first-visit probabilities, $\sum_{j=0}^{\infty} f(r, \tau_j) = 1$.

Equation (1) is the *first visit equation* [1,2] which is the analog of Einstein's master equation for the classical random walk wherefrom the usual diffusion equation follows [3]. In the hydrodynamic limit the first visit equation (1) yields the *propagation–dispersion equation* [1,2]

$$\frac{\partial}{\partial r} f(r, t) + \frac{1}{c} \frac{\partial}{\partial t} f(r, t) = \frac{1}{2} \gamma \frac{\partial^2}{\partial t^2} f(r, t), \tag{2}$$

where the value of c^{-1} is given by the first moment $J_1 = \sum_{j=1}^{N} j \, p_j$, and that of γ by the second cumulant $\sum_{j=1}^{N} j^2 p_j - J_1^2 = J_2 - J_1^2$; c is the propagation speed of the particle, and γ the *time dispersion coefficient*. Equation (2) is the analog of the advection-diffusion equation, but describes a dispersion process in *time* (instead of diffusion in space) with a drift expressed by a propagation speed with non-zero bounded values. The normalization condition, $\int_0^{\infty} f(r, t) dt = 1$, requires that $\left(\frac{\partial f(r,t)}{\partial t} \right)_{t=0} = 0$ for any $r > 0$. The solution to Eq. 2 is a Gaussian in time [1,2]. A typical example of a time dispersion process is the Marathon race where one considers the distribution of the arrival times of the runners as illustrated in Fig. 1. However one nevertheless observes deviations from the Gaussian in the wings of the distribution indicating that the Gaussian is clearly an approximation.

Consider now that the waiting times depend on the particle distribution function (in the example of the marathon (Fig. 1) this corresponds to the local concentration of runners): other typical examples are the dispersion process of particles in the cellular membrane or in media subject to concentration gradients such as crowded environments in biological media. Then, the time delay probability p_j in Eq. (1) is replaced by a functional of $f(r, t)$

$$P_j \equiv p_j F_j^{(\nu)}[f] \tag{3}$$

with the normalization $\sum_j P_j = \sum_{j=1}^{N} p_j F_j^{(\nu)}[f] = 1$, and where the index ν is such that $F^{(\nu=1)}[f] = 1$. The functional plays the role of a weighting factor to the amplitude of the waiting time probabilities. $F_j^{(\nu)}[f]$ is a functional which means that, in principle, it depends on $f(r, t)$, $f(r, t - \delta t)$, $f(r, t - 2\delta t)$, …For example, if we consider an algebraic function of the form $F_j^{(\nu)}[f] \sim f^{\nu-1}(r, t - j\delta t)$, then the normalization demands

$$F_j^{(\nu)}[f] = \frac{f^{\nu-1}(r, t - j\delta t)}{\sum_{l=1}^{N} p_l f^{\nu-1}(r, t - l\delta t)}. \tag{4}$$

So for clarity, we should write explicitly

$$F_j^{(\nu)}[f] = F_j[f(r, t - j\delta t); f(r, t - \delta t), f(r, t - 2\delta t), \ldots, f(r, t - N\delta t)], \tag{5}$$

which allows for an explicit dependence on the index j. A slightly more restricted form which does not include the j dependence, i.e. $F_j[f_j; f_1, \ldots, f_N] = F[f_j; f_1, \ldots, f_N]$ will be considered below. With (3), Eq. (1) becomes the *generalised recurrence relation*

$$f(r + \delta r, t) - f(r, t) = \sum_{j=1}^{n} p_j F_j^{(\nu)}[f_j] (f(r, t - j\delta t) - f(r, t)). \tag{6}$$

3 Nonlinear Fokker–Planck Equation

We consider the expansion of $F_j^{(\nu)}[f]$ (for simplicity in the notation we shall omit the upper index (ν) which will be reintroduced when necessary):

$$F_j[f] = \left\{ F_j(x; y_1 \ldots, y_N) \right\}_f$$
$$-\delta t \left\{ j \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial y_l} \right\}_f \left(\frac{\partial f(r, t)}{\partial t} \right) + \ldots, \tag{7}$$

where the notation $\{\ldots\}_f$ means that all the variables $x, y_1, \ldots$ are to be set equal to the $f(r, t)$'s as on the r.h.s. of Eq. 5. Using this expansion to second order, the generalized recurrence relation (6) becomes

$$\delta r \frac{\partial f(r, t)}{\partial r} + \frac{1}{2} (\delta r)^2 \frac{\partial^2 f(r, t)}{\partial r^2} + \ldots =$$

$$-\delta t \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_f \frac{\partial f(r, t)}{\partial t}$$

$$+\frac{1}{2} (\delta t)^2 \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_f \frac{\partial^2 f(r, t)}{\partial t^2}$$

$$+ (\delta t)^2 \sum_{j=1}^{N} j p_j \left\{ j \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial y_l} \right\}_f \left(\frac{\partial f(r, t)}{\partial t} \right)^2. \tag{8}$$

By multiscale expansion and using the normalization condition (see details in Appendix 1) we obtain

$$\frac{\partial f(r,t)}{\partial r} + J_1[f] \frac{\delta t}{\delta r} \frac{\partial f(r,t)}{\partial t}$$

$$= \left(J_2[f] - (J_1[f])^2\right) \frac{(\delta t)^2}{2\,\delta r} \frac{\partial^2 f(r,t)}{\partial t^2} + \Lambda[f] \frac{(\delta t)^2}{\delta r} \left(\frac{\partial f(r,t)}{\partial t}\right)^2 \qquad (9)$$

where the J_i's are the generalized moments

$$J_1[f] = \sum_{j=1}^{N} j p_j \left\{F_j(x; y_1 \ldots, y_N)\right\}_f ,$$

$$J_2[f] = \sum_{j=1}^{N} j^2 p_j \left\{F_j(x; y_1 \ldots, y_N)\right\}_f ,$$

$$\Lambda[f] = \sum_{j=1}^{N} j p_j \left\{j \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial y_l}\right\}_f$$

$$- \sum_{j=1}^{N} j p_j \left\{F_j(x; y_1 \ldots, y_N)\right\}_f \times$$

$$\times \sum_{l=1}^{N} l p_l \left\{\frac{\partial F_l(x; y_1 \ldots, y_N)}{\partial x} + \sum_{k=1}^{N} \frac{\partial F_l(x; y_1 \ldots, y_N)}{\partial y_k}\right\}_f . \qquad (10)$$

Equation (9) gives the general form of the *generalized temporal Fokker–Planck equation* (GFPE).

We consider the case where $F_j(x; y_1 \ldots, y_N)$ does not depend explicitly on j, i.e. $F_j(x; y_1 \ldots, y_N) = F(x; y_1 \ldots, y_N)$, in which case the normalization conditions (41) imply

$$0 = \left\{\frac{\partial F(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} \frac{\partial F(x; y_1 \ldots, y_N)}{\partial y_l}\right\}_f , \qquad (11)$$

so that the second term on the r.h.s. of $\Lambda[f]$ in (10) vanishes, and we have

$$J_1[f] = J_1 = \sum_j j p_j \quad ; \quad J_2[f] = J_2 = \sum_j j^2 p_j ,$$

$$\Lambda[f] = \left\{J_2 \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + J_1 \sum_{l=1}^{N} l \frac{\partial F(x; y_1 \ldots, y_N)}{\partial y_l}\right\}_f$$

$$= \left(J_2 - J_1^2\right) \left\{\frac{\partial F(x; y_1 \ldots, y_N)}{\partial x}\right\}_f . \qquad (12)$$

The generalized Fokker–Planck equation (9) then becomes

$$\frac{\partial f(r,t)}{\partial r} + \frac{1}{c} \frac{\partial f(r,t)}{\partial t} = \frac{1}{2} \gamma \left[\frac{\partial^2 f(r,t)}{\partial t^2} + 2 \left\{\frac{\partial F^{(\nu)}(x; y_1 \ldots, y_N)}{\partial x}\right\}_f \left(\frac{\partial f(r,t)}{\partial t}\right)^2\right],$$

$$c^{-1} = \sum_j p_j j \frac{\delta t}{\delta r} , \qquad \gamma = \left[\sum_j p_j j^2 - \left(\sum_j p_j j\right)^2\right] \frac{(\delta t)^2}{\delta r} . \qquad (13)$$

Here c and γ are the propagation speed and the temporal dispersion coefficient respectively. In Eq. (13) we have reintroduced the index ν for later discussion. Since by definition $F^{(\nu=1)}[f] = 1$, it is clear that for $\nu = 1$, Eq. (13) [as well as (9)] reduces to the usual propagation–dispersion equation (2).

4 Scaling and Power Law Distribution

We now ask for a scaling solution of the GFPE

$$f(r, t) = r^{-\beta} \phi \left(\frac{t - r/c}{r^{\beta}} \right), \tag{14}$$

which by substitution in (9) with $\zeta = \frac{t - r/c}{r^{\beta}}$ gives

$$-\phi(\zeta) - \zeta \phi'(\zeta) =$$
$$\beta^{-1} r^{1-2\beta} \frac{1}{2} \left[\left(J_2[f] - (J_1[f])^2 \right) \phi''(\zeta) + 2 r^{-\beta} \Lambda[f] \phi'^2(\zeta) \right]. \tag{15}$$

The scaling equation can be satisfied either with $\Lambda[f] = 0$ or with $\Lambda[f] \sim 1/f(r, t)$ and, in both cases $\beta = 1/2$. The first case is realized for $\nu = 1$, i.e. $F^{(\nu)} = 1$; then $\left(J_2[f] - (J_1[f])^2 \right) (\delta t)^2/\delta r = \gamma$ and $J_1[f] \, \delta t/\delta r = c^{-1}$, and Eq. (9) reduces to the classical temporal Fokker–Planck equation with Gaussian solution [1,2]. The second case can be satisfied if we require that $F^{(\nu)}[f]$ be a normalized power law independent of j. Indeed when $F^{(\nu)}[f]$ does not dependent on j, $\Lambda[f]$ (12) reduces to $\left(J_2 - (J_1)^2 \right) \left\{ \frac{\partial F^{(\nu)}(x; y_1 ..., y_N)}{\partial x} \right\}_f$, and from the normalization condition, one can write F in terms of an unknown function of a single variable, $K(x)$, as

$$F^{(\nu)}(x; y_1 \ldots, y_N) = \frac{K(x)}{p_1 K(y_1) + \ldots + p_N K(y_N)}, \tag{16}$$

which gives

$$\left\{ \frac{\partial F^{(\nu)}(x; y_1 \ldots, y_N)}{\partial x} \right\}_f = \frac{1}{p_1 + \ldots + p_N} \frac{1}{K(f)} \frac{\partial K(f)}{\partial f} = \frac{\partial \ln K(f)}{\partial f}. \tag{17}$$

The demand $\Lambda[f] \sim 1/f(r, t)$ implies that, for some constant $\nu \geq 1$

$$\frac{\partial \ln K(f)}{\partial f} = \frac{\nu - 1}{f} \implies K(f) = f^{\nu-1}, \tag{18}$$

so that

$$F^{(\nu)}(x; y_1 \ldots, y_N) = \frac{x^{\nu-1}}{p_1 y_1^{\nu-1} + \ldots + p_N y_N^{\nu-1}}. \tag{19}$$

Notice that the power law form follows from the scaling. It is not being introduced as an arbitrary ansatz but, rather, is the only functional form that satisfies the requirement of producing diffusive behavior. With this result and with $\beta = 1/2$, Eq. (15) becomes

$$-\phi(\zeta) - \zeta \phi'(\zeta) = \gamma \left[\phi''(\zeta) + 2(\nu - 1) \phi^{-1}(\zeta) \phi'^2(\zeta) \right]. \tag{20}$$

We also note that if one uses (19) as an *ansatz* in the GFPE [Eq. (13)] one obtains

$$\frac{\partial f(r, t)}{\partial r} = \frac{1}{2} \gamma \left[\frac{\partial^2 f(r, t)}{\partial t^2} + 2 \frac{\nu - 1}{f(r, t)} \left(\frac{\partial f(r, t)}{\partial t} \right)^2 \right], \tag{21}$$

which with the scaling relation (14) gives exactly (20).

Equation (20) can be simplified by introducing a change of variables and a transformation

$$x = \zeta/\sqrt{\gamma} = \frac{t - r/c}{\sqrt{\gamma\,r}} \quad ; \quad \phi\,(\zeta) = w^{\frac{1}{2\nu-1}}\,(x)\,, \tag{22}$$

giving

$$\frac{d^2 w}{dx^2} + x\frac{dw}{dx} + (2\nu - 1)\,w = 0, \tag{23}$$

which equation can be matched to the general confluent equation (see Appendix 2) and has the general solution

$$w\,(x) = A\exp\left(-\frac{x^2}{2}\right) M\left(1 - \nu, \frac{1}{2}, \frac{x^2}{2}\right) + Bx\exp\left(-\frac{x^2}{2}\right) M\left(\frac{3}{2} - \nu, \frac{3}{2}, \frac{x^2}{2}\right) \tag{24}$$

where A and B are constants and $M\,(a, b, x)$ is the confluent hypergeometric function. Since the solution must be even in x, B must be zero for symmetrical reasons. So the scaling distribution reads

$$f\,(r, t) = \frac{1}{\sqrt{\gamma\,r}}\,\phi\left(\frac{t - r/c}{\sqrt{\gamma\,r}}\right)$$

$$= \frac{1}{\sqrt{\gamma\,r}}\left[A\,M\left(1 - \nu, \frac{1}{2}, \frac{(t - r/c)^2}{2\,\gamma\,r}\right)\exp\left(-\frac{(t - r/c)^2}{2\,\gamma\,r}\right)\right]^{\frac{1}{2\nu-1}}. \tag{25}$$

For $\nu = 1$, $M(0, \frac{1}{2}, \frac{x^2}{2}) = 1$; consequently in order to retrieve the normalized Gaussian solution, A must be $\sqrt{2/\pi}$ and the final solution is given by

$$f\,(r, t) = \frac{1}{\sqrt{\gamma\,r}}\left[\sqrt{\frac{2}{\pi}}\,M_{(1-\nu)}\,\exp\left(-\frac{(t - r/c)^2}{2\,\gamma\,r}\right)\right]^{\frac{1}{2\nu-1}}, \tag{26}$$

where $M_{(1-\nu)} = M(1 - \nu, \frac{1}{2}, \frac{x^2}{2})$ with $x = \frac{t-r/c}{\sqrt{\gamma\,r}}$. Figure 2 illustrates this result for different values of the exponent.

For $\nu = 3/2$, we have $M(-\frac{1}{2}, \frac{1}{2}, \frac{x^2}{2}) = \frac{1}{\sqrt{2}}\exp(\frac{x^2}{4})\,E_1^{(0)}(x)$ (where $E_1^{(0)}(x)$ is the parabolic cylinder function) giving

$$f_{\nu=3/2}\,(r, t) = \frac{1}{\sqrt{\gamma\,r}}\left(\frac{1}{\sqrt{\pi}}\,E_1^{(0)}\left(\frac{t - r/c}{\sqrt{\gamma\,r}}\right)\exp-\left(\frac{t - r/c}{2\sqrt{\gamma\,r}}\right)^2\right)^{1/2}. \tag{27}$$

For $\nu > 3/2$, we have $M(-\frac{m}{2}, \frac{1}{2}, \frac{x^2}{2})$, but the confluent hypergeometric function with $m > 1$ exhibits alternating positive and negative regions separated by a singularity and consequently so for the distribution function; therefore values of $\nu > 3/2$ must be physically rejected and the meaningful range of the exponent is $1 \le \nu \le 3/2$, as illustrated in Fig. 2. So when the nonlinear exponent ν increases we observe a narrowing of the distribution function that is a localization in temporal dispersion.

The asymptotic behavior of the distribution follows from the observation that for $|x|$ large (see AS 13.5.1 in [8])

$$M\,(a, b, x) = \frac{\Gamma(b)}{\Gamma(a)}\,e^x\,x^{a-b}\left(1 + \mathcal{O}(|x|^{-1})\right) \quad ; \quad x \in \Re, > 0, \tag{28}$$
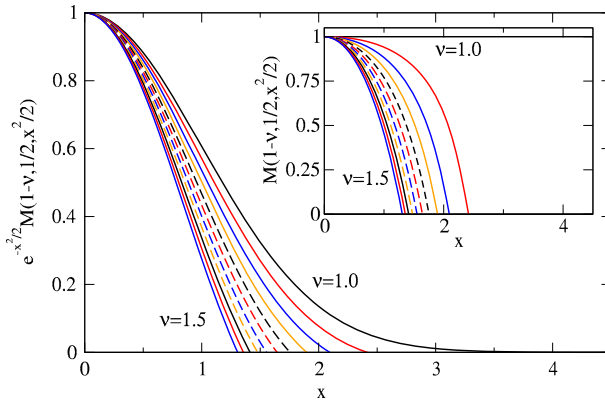
Fig. 2 $M(1 - \nu, \frac{1}{2}, \frac{x^2}{2}) \exp\left(-\frac{x^2}{2}\right)$ [Eq. (28)] as a function of x for various values of the exponent $1 \leq \nu \leq 3/2$. The *inset* shows the coefficient of the Gaussian term which for $\nu = 1.0$ is one

which when used in (26) gives

$$f(x \gg 1) \simeq \frac{1}{\sqrt{\gamma \, r}} \left(\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1 - \nu)} \left(\frac{x^2}{2}\right)^{\frac{1-2\nu}{2}} \right)^{\frac{1}{2\nu-1}} \tag{29}$$

or, with $t \gg r/c$,

$$f\left(r, t \gg \sqrt{\gamma \, r}\right) \simeq \left(\frac{1}{\Gamma(1 - \nu)} \right)^{\frac{1}{2\nu-1}} \frac{\sqrt{2}}{t}, \tag{30}$$

that is, for long times, $f(r, t) \sim t^{-1}$, which is in reasonable agreement with the observation of time delays in earthquake distributions [9,10] $P(t) \sim t^{-\gamma}$ with $\gamma \simeq 0.9$.

5 Fractional Fokker–Planck Equation

The generalization of temporal diffusion to nonlinear waiting time probabilities discussed so far was developed based on a multiscale expansion that is only valid when the first and second moments of the waiting time probability exist. We now consider a second generalization that applies when the second moment does not exist. Unlike the nonlinear case, we will only consider processes for which the waiting time probabilities are statistically independent. When the second moment exists, this then leads via the central limit theorem to the classical Gaussian time distribution (see Fig. 1) since for the lattice model described in Sect. 2 the probability to reach lattice position l in time is simply the sum of the independent random waiting times $\widehat{t} = \sum_{l=1}^{n} \widehat{t_l}$. Explicitly, for large l, that is $r \gg \delta r$, the probability for the stochastic variable $\widehat{t}$ to lie in the interval $[T, T + dT]$ is given by a Gaussian

$$f(T; r) = \sqrt{\frac{1}{2\pi\sigma}} \exp\left(-\frac{\left(T - \bar{t}(r)\right)^2}{2\sigma}\right), \tag{31}$$

where the most likely time is

$$\bar{t} = \sum_{l=1}^{n} \langle \widehat{t_l} \rangle = n \int_0^\infty d\tau \ \tau p \ (\tau) \tag{32}$$

with $p(\tau)$, the time delay probability, and the width of the distribution is

$$\sigma(r) = \sum_{l=1}^{n} \left(\langle \widehat{t_l}^2 \rangle - \langle \widehat{t_l} \rangle^2 \right) = n \int_0^\infty d\tau \ \tau^2 p(\tau) - \frac{1}{n} \bar{t}^2. \tag{33}$$

Consider now the case of distributions which do not possess second moments. In particular, we consider a power law distribution

$$p(t) = \frac{\alpha t_0^\alpha}{t^{\alpha+1}} \Theta(t - t_0) \quad ; \quad 0 < \alpha < 2 \quad ; \quad t \to \infty, \tag{34}$$

and let $\widehat{t} = \frac{1}{N^{1/\alpha}} \sum_{l=1}^{n} \widehat{t_l}$. In this case, straightforward analysis, see Appendix 3 for details, results in the *fractional temporal diffusion equation*

$$\frac{\partial}{\partial r} f(T; r/\delta r) = \left[\frac{\alpha t_0}{(1-\alpha)\delta r} \frac{\partial}{\partial T} + \frac{\alpha t_0^\alpha}{\delta r} \Gamma(-\alpha) \frac{\partial^\alpha}{\partial T^\alpha} \right] f_\alpha(T; r/\delta r). \tag{35}$$

Notice that, with time and space variables interchanged, the FFPE exhibits a structure analogous to the fractional Fokker–Planck equation for anomalous spatial diffusion that follows from the continuous time random walk model with a power law *ansatz* for the waiting times [6].

6 Conclusions

When considering diffusion processes from the viewpoint of a temporal formulation—dual to the classical spatial description—a Fokker–Planck equation (FPE) description is found to be equally valid for temporal diffusion. In the latter case the FPE exhibits a solution for the temporal distribution function showing Gaussian behavior [1,2] similar to the Gaussian solution of the classical diffusion equation, but with time and space reversed. However when, as in most real systems, the diffusive medium is inhomogeneous, this classical description is modified because the dynamics, and consequently the corresponding distribution function, may depend on the local concentration variations in time and space and on the distribution of time delays in the diffusive process. We considered both types of dependencies. (i) Starting from the classical random walk model, we generalized an Einstein-like recurrence relation by including a functional concentration dependence in the waiting time probability wherefrom a temporal nonlinear Fokker–Planck equation is obtained and solved to yield the temporal distribution function evolving from Gaussian shape to finite support when the nonlinear exponent increases. (ii) On the other hand using a power law waiting time probability distribution we obtain a fractional temporal Fokker–Planck equation similar to the usual fractional Fokker–Planck equation [6] with space and time interchanged. These results should provide insight for the elucidation of the mechanisms of temporal diffusion processes. Our analysis shows how certain microscopic mechanisms, e.g. weighting times that are influenced by the local density of random walkers, can lead to non-Gaussian distributions with finite support and non-classical scaling exponents. In particular, our results show that a non-Gaussian power-law distribution of first passage times follows either straightforwardly from an ansatz which is shown to be the solution of the fractional equation or from the asymptotic behavior

of the solution of the generalized Fokker–Planck equation (FPE) with the distinction that with the power law ansatz the exponent is merely a parameter whereas in the FPE approach the asymptotic power law expression has no adjustable parameter and the exponent is obtained analytically [$\sim t^{-1}$, Eq. 30]

Finally, we note that the solutions to the generalized FPE bear some similarity to those found for certain self-similar Fokker–Planck equations [11] so that one might expect a connection between the two. In fact, the latter case would, in our language, correspond to a FPE with time-dependent drift velocity and diffusion coefficient whereas in our case those quantities actually depend on the distribution itself. So any connection between the two models is probably fortuitous.

Appendix 1: Expansion of the Recurrence Relation

We first consider the expansion of $F_j^{(\nu)}[f]$ (for simplicity in the notation we shall omit the upper index (ν) which will be reintroduced when necessary):

$$
F_j[f] = \left\{ F_j(x; y_1 \dots, y_N) \right\}_f
$$
$$
- \delta t \left\{ j \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial y_l} \right\}_f \times \left(\frac{\partial f(r,t)}{\partial t} \right) + \dots \quad (36)
$$

where the notation $\{\dots\}_f$ means that all the variables $x, y_1, \dots$ are to be set equal to the $f(r,t)$'s in the r.h.s. of Eq. (5). Using this expansion, the generalized recurrence relation (6) becomes

$$
\delta r \frac{\partial f(r,t)}{\partial r} + \frac{1}{2} (\delta r)^2 \frac{\partial^2 f(r,t)}{\partial r^2} + \dots =
$$
$$
- \delta t \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \dots, y_N) \right\}_f \frac{\partial f(r,t)}{\partial t}
$$
$$
+ \frac{1}{2} (\delta t)^2 \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \dots, y_N) \right\}_f \frac{\partial^2 f(r,t)}{\partial t^2}
$$
$$
+ (\delta t)^2 \sum_{j=1}^{N} j p_j \left\{ j \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial y_l} \right\}_f \left(\frac{\partial f(r,t)}{\partial t} \right)^2 + \dots
$$
$$
(37)
$$

We now perform a multiscale expansion with

$$
\frac{\partial}{\partial r} \to \epsilon \frac{\partial}{\partial r_1} + \epsilon^2 \frac{\partial}{\partial r_2} \quad ; \quad \frac{\partial}{\partial t} \to \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}, \quad (38)
$$

and $f = f^{(0)} + \epsilon f^{(1)} + \mathcal{O}(\epsilon^2)$, where $f^{(0)}$ is the distribution function in the absence of dispersion. To first order, we obtain

$$
\mathcal{O}(\epsilon^1): \quad \delta r \frac{\partial f^{(0)}(r,t)}{\partial r_1} = -\delta t \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \dots, y_N) \right\}_{f^{(0)}} \frac{\partial f^{(0)}(r,t)}{\partial t_1}, \quad (39)
$$

and to second order

$$\mathcal{O}(\epsilon^2): \quad \delta r \frac{\partial f^{(1)}(r,t)}{\partial r_1} + \delta r \frac{\partial f^{(0)}(r,t)}{\partial r_2} + \frac{1}{2}(\delta r)^2 \frac{\partial^2 f^{(0)}(r,t)}{\partial r_1^2}$$

$$= -\delta t \sum_{j=1}^{N} jp_j \left\{ \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial x} + \sum_{l=1}^{N} \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial y_l} \right\}_{f^{(0)}} f^{(1)}(r,t) \frac{\partial f^{(0)}(r,t)}{\partial t_1}$$

$$-\delta t \sum_{j=1}^{N} jp_j \{F_j(x; y_1 \dots, y_N)\}_f \left(\frac{\partial f^{(1)}(r,t)}{\partial t_1} + \frac{\partial f^{(0)}(r,t)}{\partial t_2} \right)$$

$$+\frac{1}{2}(\delta t)^2 \sum_{j=1}^{N} j^2 p_j \{F_j(x; y_1 \dots, y_N)\}_f \frac{\partial^2 f^{(0)}(r,t)}{\partial t_1^2}$$

$$+(\delta t)^2 \sum_{j=1}^{N} jp_j \left\{ j \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial y_l} \right\}_{f^{(0)}} \left(\frac{\partial f^{(0)}(r,t)}{\partial t_1} \right)^2. \tag{40}$$

From the normalization condition (with (7) where $\frac{\partial f(r,t)}{\partial t}$ is unconstrained)

$$1 = \sum_{j=1}^{N} p_j \{F_j(x; y_1 \dots, y_N)\}_{f^{(0)}},$$

$$0 = \sum_{j=1}^{N} p_j \left\{ j \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \dots, y_N)}{\partial y_l} \right\}_{f^{(0)}}. \tag{41}$$

It is easy, for instance, to check that these relations are indeed verified in the case of the power law (4). Differentiating (39) with respect to r_1 and reinserting (39) in the result, we obtain

$$(\delta r)^2 \frac{\partial^2 f^{(0)}(r,t)}{\partial r_1^2} = -\delta r \delta t \sum_{j=1}^{N} jp_j \{F_j(x; y_1 \dots, y_N)\}_{f^{(0)}} \frac{\partial^2 f^{(0)}(r,t)}{\partial t_1 \partial r_1}$$

$$= (\delta t)^2 \sum_{j=1}^{N} jp_j \{F_j(x; y_1 \dots, y_N)\}_{f^{(0)}} \frac{\partial}{\partial t_1} \left\{ \sum_{l=1}^{N} lp_l F_l(x; y_1 \dots, y_N) \right\}_{f^{(0)}} \frac{\partial f^{(0)}(r,t)}{\partial t_1}$$

$$= (\delta t)^2 \sum_{j=1}^{N} jp_j \{F_j(x; y_1 \dots, y_N)\}_{f^{(0)}} \times$$

$$\times \sum_{l=1}^{N} lp_l \left\{ \frac{\partial F_l(x; y_1 \dots, y_N)}{\partial x} + \sum_{k=1}^{N} \frac{\partial F_l(x; y_1 \dots, y_N)}{\partial y_k} \right\}_{f^{(0)}} \left(\frac{\partial f^{(0)}(r,t)}{\partial t_1} \right)^2$$

$$+(\delta t)^2 \left(\sum_{j=1}^{N} jp_j \{F_j(x; y_1 \dots, y_N)\}_{f^{(0)}} \right)^2 \frac{\partial^2 f^{(0)}(r,t)}{\partial t_1^2}. \tag{42}$$

Using this result in (40) gives

$$\delta r \frac{\partial f^{(1)}(r,t)}{\partial r_1} + \delta r \frac{\partial f^{(0)}(r,t)}{\partial r_2}$$

$$= -\delta t \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_{f^{(0)}} \left(\frac{\partial f^{(1)}(r,t)}{\partial t_1} + \frac{\partial f^{(0)}(r,t)}{\partial t_2} \right)$$

$$- \delta t \sum_{j=1}^{N} j p_j \left\{ \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial y_l} \right\}_{f^{(0)}} f^{(1)}(r,t) \frac{\partial f^{(0)}(r,t)}{\partial t_1}$$

$$+ \frac{1}{2} (\delta t)^2 \left(\sum_{j=1}^{N} j^2 p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_{f^{(0)}} - \left(\sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_{f^{(0)}} \right)^2 \right) \frac{\partial^2 f^{(0)}(r,t)}{\partial t_1^2}$$

$$+ (\delta t)^2 \left(\begin{array}{c} \sum_{j=1}^{N} j p_j \left\{ j \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial x} + \sum_{l=1}^{N} l \frac{\partial F_j(x; y_1 \ldots, y_N)}{\partial y_l} \right\}_{f^{(0)}} \\ - \sum_{j=1}^{N} j p_j \left\{ F_j(x; y_1 \ldots, y_N) \right\}_{f^{(0)}} \sum_{l=1}^{N} l p_l \left\{ \frac{\partial F_l(x; y_1 \ldots, y_N)}{\partial x} + \sum_{k=1}^{N} \frac{\partial F_l(x; y_1 \ldots, y_N)}{\partial y_k} \right\}_{f^{(0)}} \end{array} \right)$$

$$\times \left(\frac{\partial f^{(0)}(r,t)}{\partial t_1} \right)^2, \tag{43}$$

After recombining first and second order terms, resummation yields Eq. (9).

Appendix 2: General Confluent Equation

$$\frac{d^2 w}{dx^2} + x \frac{dw}{dx} + (2v - 1) w = 0, \tag{44}$$

From AS 13.1.35 in [8], the general confluent equation is

$$0 = w'' + \left(\frac{2a}{y} + 2f'(y) + \frac{bh'(y)}{h(y)} - h'(y) - \frac{h''(y)}{h'(y)} \right) w'$$

$$+ \left(\frac{bh'(y)}{h(y)} - h'(y) - \frac{h''(y)}{h'(y)} \right) \left(\frac{a}{y} + f'(y) \right) w$$

$$+ \left(\frac{a(a-1)}{y^2} + \frac{2af'(y)}{y} + f''(y) + f'^2(y) - \frac{ch'^2(y)}{h(y)} \right) w. \tag{45}$$

This equation and Eq. (44) match provided

$$a = 0 \; ; \quad b = \frac{1}{2} \; ; \quad c = 1 - v \; ; \quad f(y) = h(y) = \frac{y^2}{2},$$

and the general solution then is

$$w(y) = A \exp\left(-\frac{y^2}{2} \right) M\left(1 - v, \frac{1}{2}, \frac{y^2}{2} \right) + B \exp\left(-\frac{y^2}{2} \right) U\left(1 - v, \frac{1}{2}, \frac{y^2}{2} \right), \tag{46}$$

or, using AS 13.1.3 in [8], with $y \equiv x$,

$$w(x) = A \left[M\left(1 - v, \frac{1}{2}, \frac{x^2}{2} \right) + x \frac{B}{A} M\left(\frac{3}{2} - v, \frac{3}{2}, \frac{x^2}{2} \right) \right] \exp\left(-\frac{x^2}{2} \right). \tag{47}$$

Note that $w(0) = A$ and $w'(0) = B$.

Appendix 3: Derivation of the Fractional Temporal Diffusion Equation

The probability for $\hat{t} = T_\alpha$ is

$$f_\alpha(T_\alpha, N) = \int_0^\infty \delta(T_\alpha - \frac{1}{N^{1/\alpha}} \sum_{i=1}^N t_i) p(t_1) \ldots p(t_N) dt_1 \ldots dt_N \quad (48)$$

so that

$$\tilde{f}_\alpha(\omega, N) \equiv \int_{-\infty}^\infty e^{i\omega T} f_\alpha(T_\alpha) = \left(\tilde{p} \left(\frac{\omega}{N^{1/\alpha}} \right) \right)^N, \quad (49)$$

where a simple calculation gives

$$\tilde{p}(\omega) = \alpha(-i\omega t_0)^\alpha \Gamma(-\alpha, -i\omega t_0), \quad (50)$$

which has the expansion for small ωt_0

$$\tilde{p}(\omega) = \alpha(-i\omega t_0)^\alpha \Gamma(-\alpha) - \alpha \sum_{k=0}^\infty \frac{(-i\omega t_0)^k}{k!(k-\alpha)}. \quad (51)$$

Thus, taking the inverse Fourier transform and expanding in $1/N$ leads to the result

$$f_\alpha(T_\alpha, N) = \int_{-\infty}^\infty \left(\tilde{p} \left(\frac{\omega}{N^{1/\alpha}} \right) \right)^N e^{-i\omega T_\alpha} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^\infty \exp \left(-i\omega T_\alpha + N \ln \tilde{p} \left(\frac{\omega}{N^{1/\alpha}} \right) \right) \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^\infty \exp \left(-i\omega T_\alpha + \alpha(-i\omega t_0)^\alpha \Gamma(-\alpha) + \frac{\alpha}{1-\alpha}(-i\omega t_0)N^{1-1/\alpha} \right)$$

$$\times \exp \left(\mathcal{O}(N^{1-2/\alpha}) \right) \frac{d\omega}{2\pi}, \quad (52)$$

so that the higher order terms are negligible for large N provided $\alpha < 2$. The probability density that the first arrival time to reach position N is $T = N^{1/\alpha} T_\alpha$ is therefore

$$f(T, N) = \int_0^\infty f_\alpha(T_\alpha, N)\delta(T - N^{1/\alpha} T_\alpha) dT_\alpha$$

$$= N^{-1/\alpha} \int_{-\infty}^\infty \exp \left(-i\omega N^{1-1/\alpha} \left(\frac{T}{N} + \frac{\alpha}{1-\alpha} t_0 \right) \right)$$

$$\times \exp \left(\alpha(-i\omega t_0)^\alpha \Gamma(-\alpha) + \mathcal{O}(N^{1-2/\alpha}) \right) \frac{d\omega}{2\pi}.$$

Rescaling the integration variable gives

$$f(T, N) = \int_{-\infty}^\infty exp \left(i\omega \left(T + \frac{\alpha}{1-\alpha} t_0 N \right) + \alpha(i\omega t_0)^\alpha \Gamma(-\alpha)N \right) \frac{d\omega}{2\pi} \quad (53)$$

which is the Levy-stable distribution with stability parameter α. Defining the spatial variable $r \equiv N\delta r$, it is easy to see that this distribution satisfies the *fractional temporal diffusion equation*

$$\frac{\partial}{\partial r} f(T; r/\delta r) = \left[\frac{\alpha t_0}{(1-\alpha)\delta r} \frac{\partial}{\partial T} + \frac{\alpha t_0^\alpha}{\delta r} \Gamma(-\alpha) \frac{\partial^\alpha}{\partial T^\alpha} \right] f_\alpha(T; r/\delta r). \quad (54)$$

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